



TITLE:

# ON THE MULTIVALENT FUNCTIONS WITH BOUNCED ARGUMENT

AUTHOR(S):

NUNOKAWA, MAMORU; CAUSEY, W.M.

---

CITATION:

NUNOKAWA, MAMORU ...[et al]. ON THE MULTIVALENT FUNCTIONS WITH BOUNCED ARGUMENT. 数理解析研究所講究録 1988, 664: 27-33

ISSUE DATE:

1988-07

URL:

<http://hdl.handle.net/2433/100640>

RIGHT:

# ON THE MULTIVALENT FUNCTIONS WITH BOUNDED ARGUMENT

MAMORU NUNOKAWA (群馬大・教育 布川護)

W. M. CAUSEY (ミシシッピ大)

## ABSTRACT

The object of the present paper is to give the starlike boundary and convex boundary of certain subclass of  $p$ -valent functions in the unit disk.

### 1. Introduction

Let  $F(p)$  be the subclass of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in  $E = \{z \mid |z| < 1\}$ .

Also let  $A(\alpha)$  denote the class of functions

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

analytic in  $E$  and satisfying

$$|\arg g(z)| \leq \pi\alpha/2 \quad (z \in E)$$

where  $\alpha > 0$ .

It is well known [3] that if  $g(z) \in A(1)$ , then

$$g(z) \prec \frac{1+z}{1-z}$$

where  $\prec$  denotes "is subordinate to" and [1] that

$$(1) \quad \left| \frac{g'(z)}{g(z)} \right| \leq \frac{2}{1-r^2}, \quad \text{for } |z| = r < 1.$$

In this paper, we need the following lemmata.

LEMMA 1. Suppose  $g(z) \in A(\alpha)$ , where  $\alpha > 0$ . Then we have

$$\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{2\alpha r}{1-r^2} \quad \text{for } |z| = r < 1.$$

PROOF. Let  $h(z) = g(z)^{1/\alpha}$ . Then we have  $h(z) \in A(1)$  and from (1) we have

$$\left| \frac{h'(z)}{h(z)} \right| = \frac{1}{\alpha} \left| \frac{g'(z)}{g(z)} \right| \leq \frac{2}{1-r^2}$$

for  $|z| = r < 1$ . This completes our proof and a proof of this result can be found in [4, 7].

If a function

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

is analytic in  $E$  and does not assume non-positive real values in  $E$ , then  $g(z) \in A(2)$  and from LEMMA 1, it follows that

$$\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{4r}{1-r^2} \quad \text{for } |z| = r < 1.$$

REMARK. Throughout the paper, all powers are meant as principal values.

LEMMA 2. Suppose that  $f(z) \in F(p)$  and

$$p + \operatorname{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > 0 \quad \text{in } E.$$

Then we have

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } E,$$

and therefore  $f(z)$  is  $p$ -valently convex in  $E$ .

A proof of this lemma can be found in [5, Theorem 1].

LEMMA 3. Suppose  $f(z) \in F(p)$  and

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \quad \text{in } E.$$

Then we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E,$$

and therefore  $f(z)$  is  $p$ -valently starlike in  $E$ .

A proof of this lemma can be found in [5, Theorem 5].

LEMMA 4. Let  $h(z) \in A(2)$ . Then we have

$$\left( \frac{1-r}{1+r} \right)^2 \leq \operatorname{Re} h(z) \leq \left( \frac{1+r}{1-r} \right)^2$$

for  $|z| = r < 2 - \sqrt{3}$ , and

$$\frac{1 - 6r^2 + r^4}{2(1 - r^2)^2} \leq \operatorname{Re} h(z) \leq \left( \frac{1+r}{1-r} \right)^2$$

for  $2 - \sqrt{3} \leq |z| = r < 1$ . Therefore we have that

$$\operatorname{Re} h(z) > 0 \quad \text{in } |z| < \sqrt{2} - 1.$$

This result is sharp. A proof can be found in [6].

## 2. Main theorems

THEOREM 1. Let  $f(z) \in F(p)$  and

$$\frac{f(z)}{z^p} \in A(\alpha), \quad \text{where } \alpha > 0.$$

Then  $f(z)$  is  $p$ -valently starlike in  $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$  and the result is sharp.

PROOF. Let  $g(z) = f(z)/z^p$ . Then  $g(z) \in A(\alpha)$  and from LEMMA 1, we have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| \frac{zf'(z)}{f(z)} - p \right| \leq \frac{2\alpha r}{1 - r^2}$$

for  $|z| = r < 1$ .

Then it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq p - \frac{2\alpha r}{1 - r^2} = \frac{p - 2\alpha r - pr^2}{1 - r^2}$$

for  $|z| = r < 1$ .

This shows that  $f(z)$  is  $p$ -valently starlike in  $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$ .

The result is sharp as seen by letting

$$f(z) = z^p \left( \frac{1+z}{1-z} \right)^\alpha.$$

COROLLARY 1. Let  $f(z) \in F(p)$  and let  $f(z)/z^p$  do not assume non-positive real values in  $E$ . Then  $f(z)$  is  $p$ -valently starlike in  $|z| < (\sqrt{4 + p^2} - 2)/p$  and the result is sharp as seen by letting

$$f(z) = z^p \left( \frac{1+z}{1-z} \right)^2.$$

Applying the same method as the proof of THEOREM 1, we have the following theorem.

THEOREM 2. Let  $f(z) \in F(p)$  and  $f^{(p)}(z)/(p!) \in A(\alpha)$ . Then  $f(z)$  is  $p$ -valently convex in  $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$ .

PROOF. Let  $g(z) = f^{(p)}(z)/(p!)$ . From LEMMA 1, we have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| \leq \frac{2\alpha r}{1-r^2}$$

for  $|z| = r < 1$ .

Then, it follows that

$$p + \operatorname{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \geq p - \frac{2\alpha r}{1-r^2} > 0$$

in  $|z| = r < (\sqrt{\alpha^2 + p^2} - \alpha)/p$ .

Therefore, from LEMMA 2, we have

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } |z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p.$$

This shows that  $f(z)$  is  $p$ -valently convex in  $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$ .

This completes our proof.

REMARK. For the case  $p = 1$  and  $\alpha = 1$ , this result is sharp [1, 4].

COROLLARY 2. Let  $f(z) \in F(p)$  and let  $f^{(p)}(z)/(p!) \in A(2)$ . Then  $f(z)$  is  $p$ -valently convex in  $|z| < (\sqrt{4 + p^2} - 2)/p$ .

THEOREM 3. Let  $f(z) \in F(p)$  and  $f^{(p-1)}(z)/(p!z) \in A(\alpha)$ .

Then  $f(z)$  is  $p$ -valently starlike in  $|z| < \sqrt{\alpha^2 + 1} - \alpha$ .

PROOF. Let  $g(z) = f^{(p-1)}(z)/(p|z)$ . Then from LEMMA 1, we easily have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| \leq \frac{2\alpha r}{1-r^2}$$

for  $|z| = r < 1$ .

It follows that

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \quad \text{in } |z| < \sqrt{\alpha^2 + 1} - \alpha.$$

Then, from LEMMA 3, we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } |z| < \sqrt{\alpha^2 + 1} - \alpha.$$

This shows that  $f(z)$  is  $p$ -valently starlike in  $|z| < \sqrt{\alpha^2 + 1} - \alpha$ .

This completes our proof.

REMARK. For the case  $p = 1$  and  $\alpha = 1$ , this result is sharp [1, 2, 4].

COROLLARY 3. Let  $f(z) \in F(p)$  and let  $f^{(p-1)}(z)/z$  do not assume non-positive real values in  $E$ . Then  $f(z)$  is  $p$ -valently starlike in  $|z| < \sqrt{5} - 2$ .

THEOREM 4. Let  $f(z) \in F(p)$  and let

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \in A(\alpha), \quad 0 < \alpha \leq 1.$$

Then  $f(z)$  is  $p$ -valently convex in  $|z| < \rho$  where  $\rho$  is the root of the equation

$$\left( \frac{1-r}{1+r} \right)^\alpha - \frac{2\alpha r}{1-r^2} = 0.$$

PROOF. Let  $g(z) = zf^{(p)}(z)/f^{(p-1)}(z)$ . Then from LEMMA 1, we have

$$\begin{aligned} \left| \frac{zg'(z)}{g(z)} \right| &= \left| 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right| \\ &\leq \frac{2\alpha r}{1-r^2} \quad \text{for } |z| = r < 1. \end{aligned}$$

Because of the assumption that  $0 < \alpha \leq 1$ , we easily have

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \geq \left( \frac{1-r}{1+r} \right)^\alpha$$

for  $|z| = r < 1$ . Then we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} &\geq \operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{2\alpha r}{1-r^2} \\ &= \left( \frac{1-r}{1+r} \right)^\alpha - \frac{2\alpha r}{1-r^2} \quad \text{for } |z| = r < 1. \end{aligned}$$

Hence we have

$$1 + \operatorname{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > 0 \quad \text{for } |z| < \rho$$

where  $\rho$  is as stated in the hypothesis of the THEOREM 4. Clearly  $0 < \rho < 1$ .

THEOREM 5. Let  $f(z) \in F(p)$  and let

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \in A(\alpha), \quad \alpha > 1.$$

Then  $f(z)$  is  $p$ -valently starlike for  $|z| < (1 - \cos(\pi/2\alpha))/\sin(\pi/2\alpha)$ .

PROOF. From the hypothesis of the Theorem, we easily have

$$\left| \arg \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right| \leq \alpha \sin^{-1} \left( \frac{2r}{1+r^2} \right)$$

for  $|z| = r < 1$ .

This shows that

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \quad \text{for } |z| < \frac{1 - \cos(\pi/2\alpha)}{\sin(\pi/2\alpha)}.$$

Therefore, from LEMMA 3 we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{for } |z| < \frac{1 - \cos(\pi/2\alpha)}{\sin(\pi/2\alpha)}.$$

This completes our proof.

COROLLARY 4. Let  $f(z) \in F(p)$  and let  $zf^{(p)}(z)/f^{(p-1)}(z) \in A(2)$ .

Then  $f(z)$  is  $p$ -valently starlike in  $|z| < \sqrt{2} - 1$ .

PROOF. From LEMMA 4, we have

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \quad \text{in } |z| < \sqrt{2} - 1.$$

## REFERENCES

- [1] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. 104(1962), 532-537.
- [2] T. H. MacGregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 104(1963), 514-520.
- [3] Z. Nehari, Conformal Mapping, McGraw-Hill, New York, 1952.
- [4] M. Nunokawa and W. M. Causey, On certain analytic functions with bounded argument, Sci. Rep. of Gurma Univ. 34(1985), 1-3.
- [5] M. Nunokawa, On the theory of multivalent functions, Tsukuba Jour. of Math. 11(2) (1987), to appear.
- [6] M. Nunokawa, S. Owa, S. Fukui, H. Saitoh and M-P. Chen, A class of functions which do not assume non-positive real values, Tamkang Jour. of Math. 19(2) (1988), to appear.
- [7] T. Yaguchi and M. Nunokawa, Functions whose derivatives do not assume non-positive real values, Proc. of the Inst. of Natur. Sci. of Nihon Univ. 23(1988), to appear.

M. Nunokawa: Department of Mathematics  
 Faculty of Education  
 Gurma University  
 Maebashi, Gurma 371  
 Japan

W. M. Causey: Department of Mathematics  
 The University of Mississippi  
 University, Ms, 38677  
 U. S. A.